Basic Structural Dynamics and Seismic Analysis

Shahram Pezeshk, Ph.D., P.E.

Associate Professor
The University of Memphis, Department of Civil Engineering,
Campus Box 526570
Memphis, TN 38152-6570
Phone: (901) 678-4727
Email: s-pezeshk@memphis.edu

References

Most of this presentation has been adopted from the following references:

Imbsen, R.A. (1981). "Seismic Design of Highway Bridges," FHWA-IP-81-2, Federal Highway Administration, January.

Imbsen, R.A. (1995). "Seismic Design of Highway Bridges," ASCE, Training Course, New York.

Chopra, A.K. (1980). **Dynamics of Structures, A Primer**, Earthquake Engineering Research institute.

Chopra, A.k. (1995). **Dynamics of Structures, Theory and Applications to Earthquake Engineering** Prentice Hall.

Chopra, A.K. "Soil and Structure Response to Earthquakes - Introduction to Structural Dynamics," EERI, Video Part I &2.

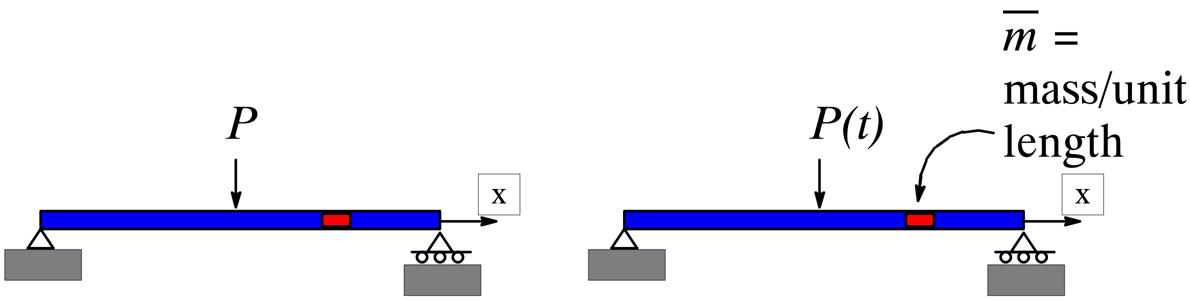
Paz, M. (1991). **Structural Dynamics, Theory and Computation.** Van Nostrand Reinhold, New York.

Characteristics of Dynamic Problems

- Characteristics of a Dynamic Problem
  - Succession of solution is required – displacement and stresses are time dependent
  - Inertia forces are part of the loading system
  - Damping forces are present – damping results in dissipation of motion
- Characteristics of a Static Problem
  - Loads are time independent
  - Magnitude of load is independent of the response mechanism

Reference Imbsen (1995)

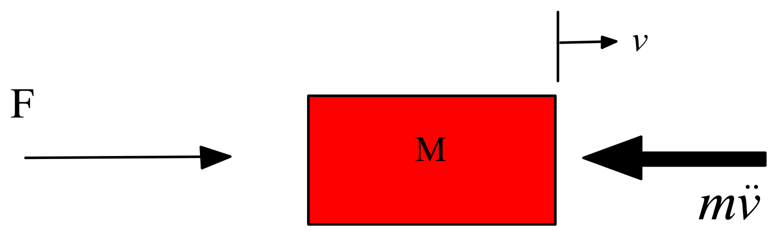
Dynamic vs. Static



- Dynamic
  - Resulting displacements are associated with accelerations which produce inertia forces resisting the acceleration
- Static
  - Structural responses are function of the applied loading and are time independent

Reference Imbsen (1995)

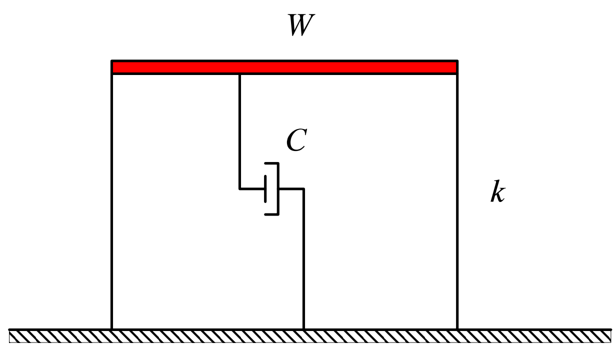
D'Alambert's Principle

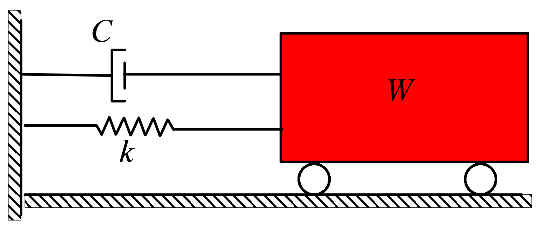


A system may be set in a state of dynamic equilibrium by adding to the external forces a fictitious force which is commonly known as the inertia force

Reference Imbsen (1995)

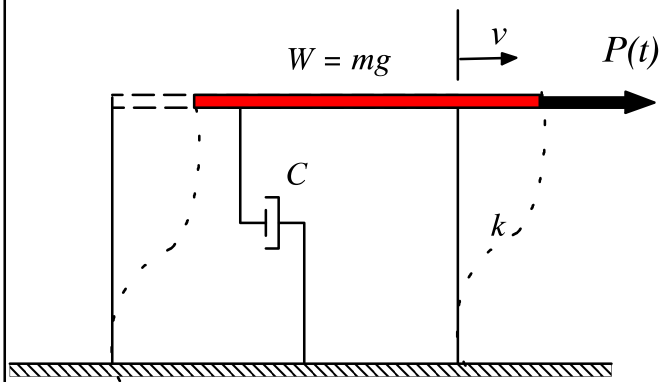
Single-Degree-of-Freedom System

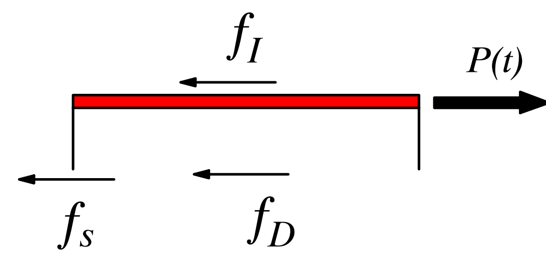


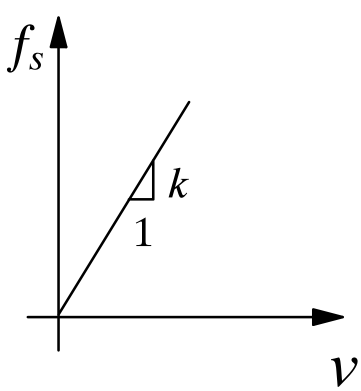


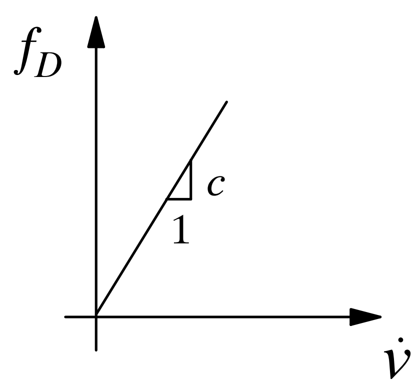
m = mass
k = stiffness of massless columns
c = coefficient of viscous damping

Idealized 1-Story Building







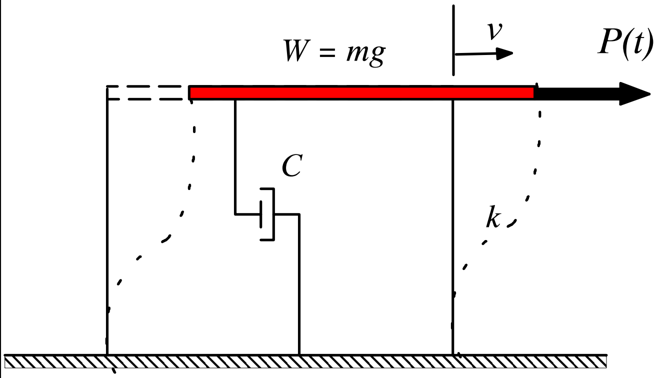


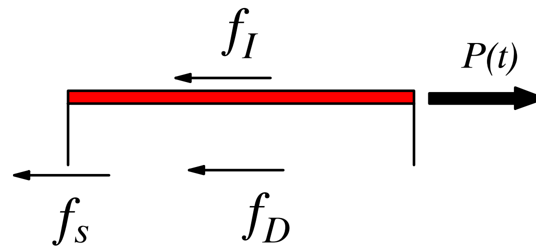
v = displacement

$\dot{v}$ = Velocity = $\frac{dv}{dt}$

$\ddot{v}$ = Acceleration = $\frac{d^2v}{dt^2}$

Idealized 1-Story Building





$$f_I + f_D + f_s = p(t)$$

$$m\ddot{v} + c\dot{v} + kv = p(t)$$

where

f_I = *Inertia Force of the Mass*

f_D = *Damping Force Acting on the Mass*

f_S = *Elastic Force*

Undamped Free Vibration

- Structure vibrates if given an initial excitation
- Damping, $c = 0$ and externally applied load, P(t) = 0,

$$m\ddot{v} + kv = 0$$

or

$$\ddot{v} + \frac{k}{m}v = 0$$

Define

$$\omega^2 = \frac{k}{m}$$

$$\omega = \sqrt{\frac{k}{m}} = \textit{Natural frequency of the system}$$

Undamped Free Vibration

Given these initial conditions

Initial displacement = $v(t=0) = v_0$

Initial velocity = $\dot{v}(t=0) = \dot{v}_0$

The final solution to the differential equation

$$\ddot{v} + \frac{k}{m}v = 0$$

$$\ddot{v} + \omega^2 v = 0$$

becomes

$$v(t) = \frac{\dot{v}_0}{\omega}\sin\omega t + v_0\cos\omega t$$

Definitions

- $$\omega = \sqrt{\frac{k}{m}}$$

  Natural frequency of the system (rad/sec)

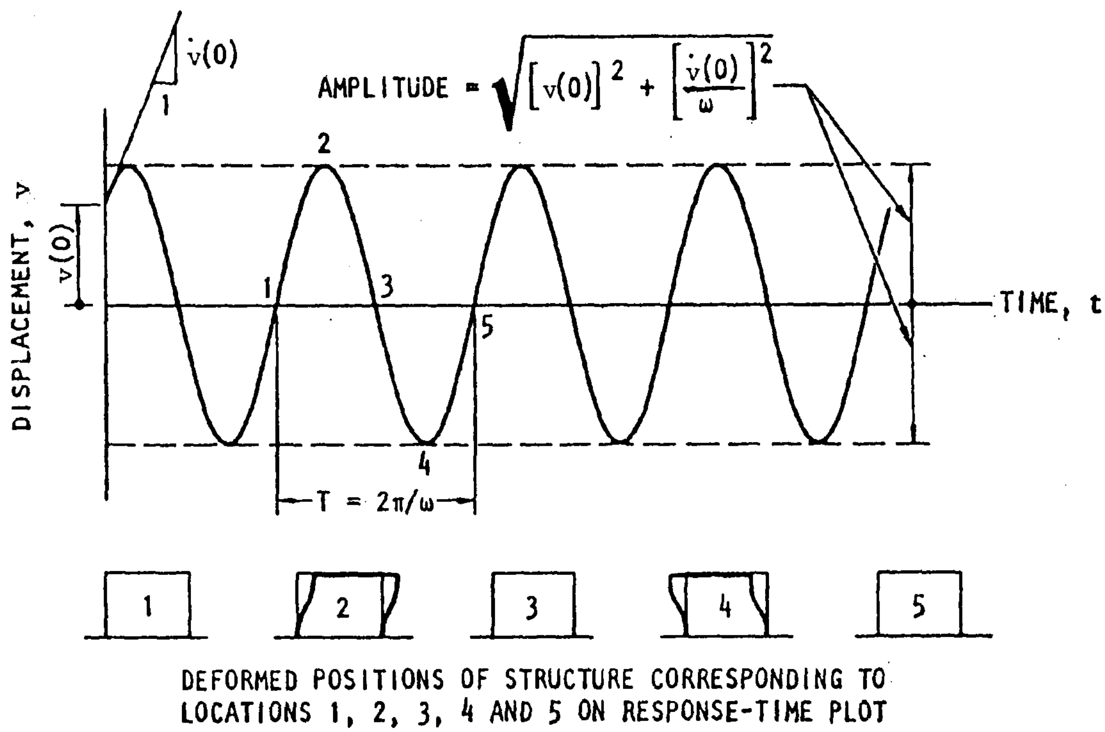
- $$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}$$

  Natural period of vibration (sec)

- $$f = \frac{\omega}{2\pi} = 2\pi\sqrt{\frac{k}{m}}$$

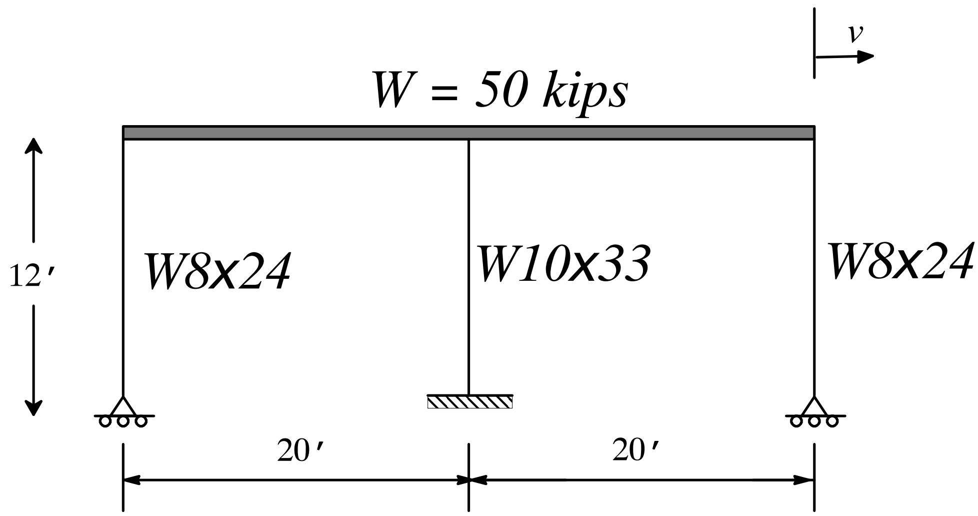
  Natural cyclic frequency of the system (1/sec or Hz)

Undamped Free Vibration



Adopted from Chopra (1980)

Example Mass, Stiffness, and Frequency Determination of a SDOF

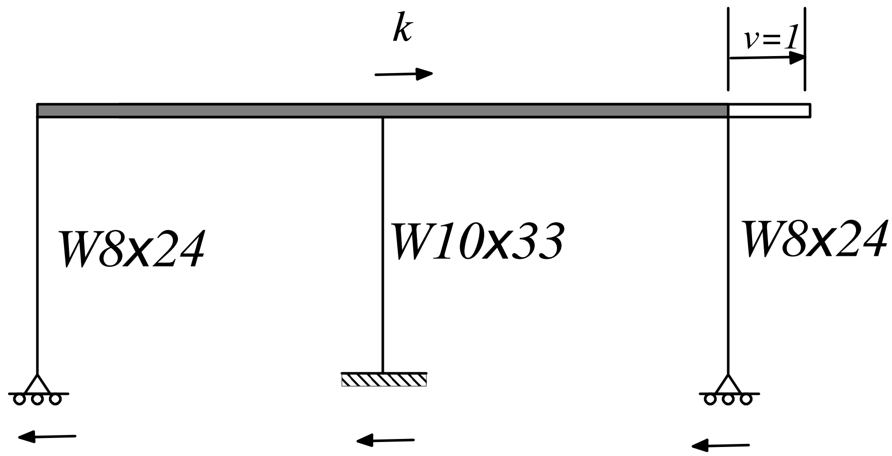


Properties

W10x33	W8x24
$A = 9.71\ in^2$	$A = 7.08\ in^2$
$I = 170\ in^4$	$I = 82.8\ in^4$

Example from Paz (1991)

Example



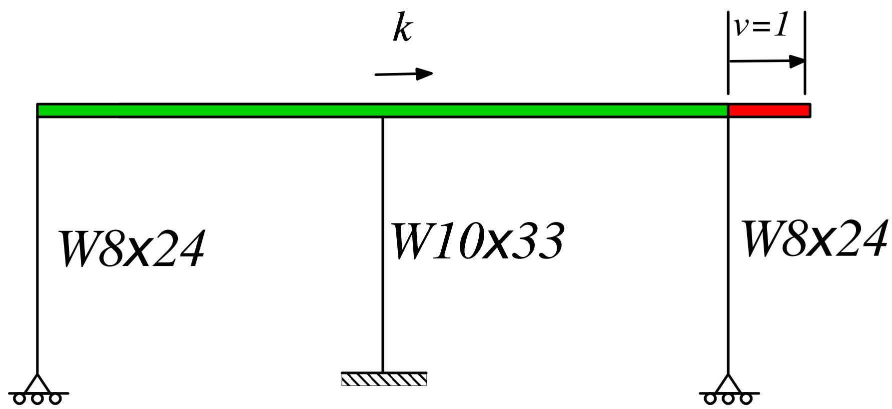
$$k_1 = \frac{3EI}{L^3} \qquad k_2 = \frac{12EI}{L^3} \qquad k_3 = \frac{3EI}{L^3}$$

- $$k_1 = k_3 = \frac{3EI}{L^3}$$
  $$k_1 = k_3 = \frac{3 \times (82.8\ in^{4}) \times (29{,}000\ k/in^2)}{144^3\ in^3}$$
  $$k_1 = k_3 = 2.4\ k/in$$

- $$k_2 = \frac{12EI}{L^3}$$
  $$k_2 = \frac{12 \times (170\ in^{4}) \times (29{,}000\ k/in^2)}{144^3\ in^3}$$
  $$k_2 = 19.8\ k/in$$

- $$k = 2 \times k_1 + k_2 = 24.6\ k/in$$

Example



- Mass

$$m = W/g = 50/386.4 = 0.129 \ \frac{k - sec^2}{in}$$

- Natural frequency of the system

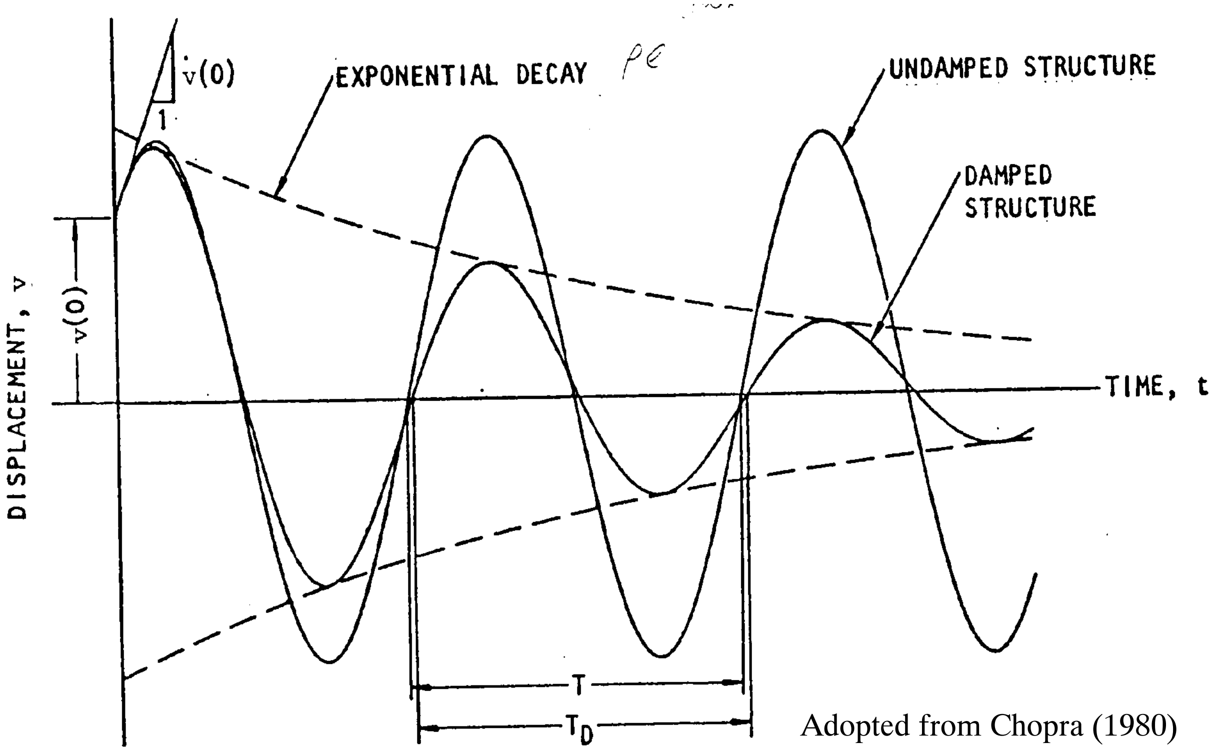
$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{24.6 \ k/in}{0.129 \ k\text{-}sec\text{^}2 \ /in}}$$

$$\omega = 13.8 \ rad/sec$$

- Period of the system

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{13.8} = 0.45 \text{ sec}$$

Damped Free Vibration



Adopted from Chopra (1980)

T_D = Natural period of damped structure

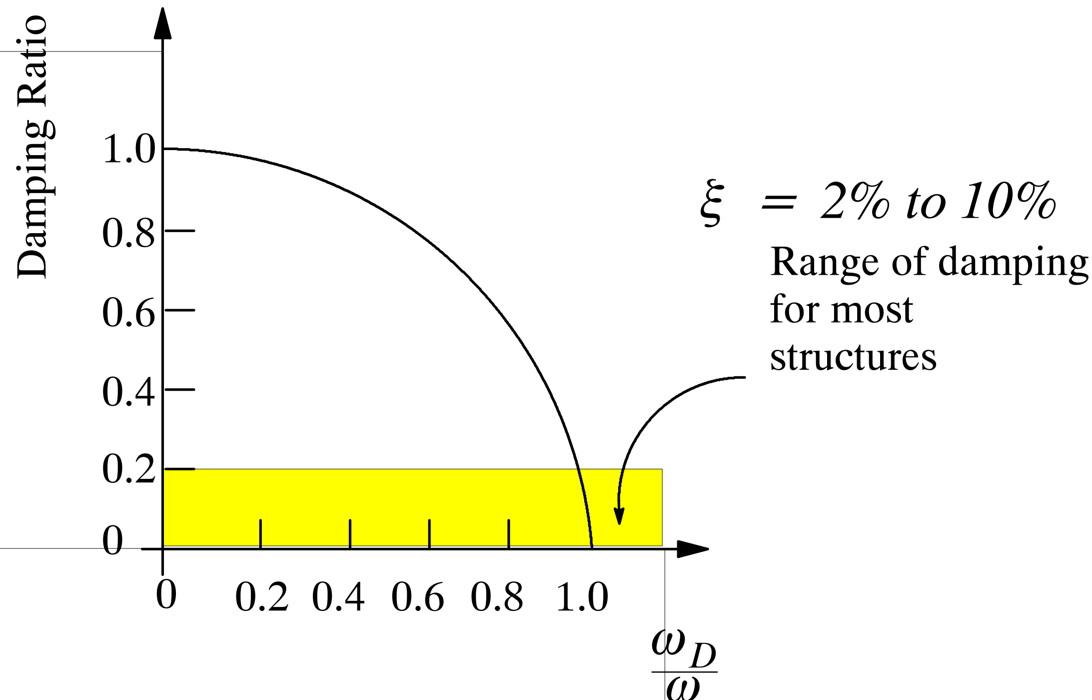
ω_D = Circular frequency of damped structure

f_D = Cyclic frequency of damped structure

$$\omega_D = \omega\sqrt{1 - \xi^2}$$

$$T_D = \frac{T}{\sqrt{1 - \xi^2}}$$

Damped Free Vibration



Reference Chopra (1980)

Sources of Damping

- Viscous damping is proportional to the magnitude of the velocity and acts opposite to the direction of motion.
  - Internal friction of material
  - Bodies moving through fluids, such as air at low velocities

Damped Free Vibration of A SDOF Structure

- Structure vibrates if given an initial excitation
- Externally applied load P(t) = 0,

$$m\ddot{v} + c\dot{v} + kv = 0$$

Solution is of the form

$$v(t) = Ge^{pt}$$

$$mGp^2e^{pt} + cGpe^{pt} + kGe^{pt} = 0$$

$$\left[mp^2 + cp + k\right]Ge^{pt} = 0$$

$$\left[mp^2 + cp + k\right] = 0$$

$$p = -\frac{c}{2m} \pm \sqrt{\left[\frac{c}{2m}\right]^2 - \frac{k}{m}}$$ *Characteristic Equation*

Damped Free Vibration of A SDOF Structure

$$\left[\frac{c}{2m}\right]^2 - \frac{k}{m} = \begin{cases} > 0 \\ = 0 \\ < 0 \end{cases}$$

Oscillatory Motion

- Critically Damped Systems
  - Minimum amount damping required to prevent structure from oscillating

$$\left[\frac{c}{2m}\right]^2 - \frac{k}{m} = 0$$

$$c_{cr} = 2m\sqrt{\frac{k}{m}} = 2\sqrt{km}$$

$$c_{cr} = 2m\sqrt{\frac{k}{m}} = 2m\omega$$

$$c_{cr} = \textit{critical damping coefficient}$$

Damping Ratio

$$c_{cr} = \textit{critical damping coefficient}$$

$$c_{cr} = 2m\sqrt{\frac{k}{m}} = 2m\omega$$

- Critical damping ratio

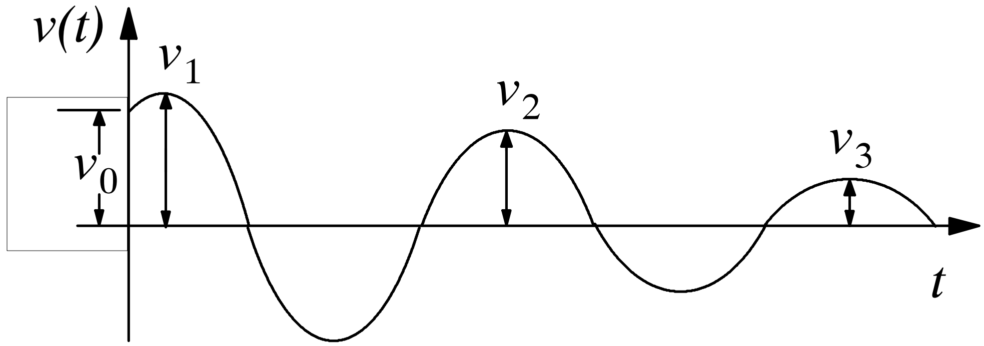
$$\xi = \frac{c}{c_{cr}} = \frac{c}{2mw}$$

$$\xi = \textit{critical damping ratio}$$

  - Damping ratio is determined experimentally

Evaluation of Damping Logarithmic Decrement

- Consider the damped free vibration test



$$\delta = \ln\left[\frac{v_1}{v_2}\right] = 2\pi\xi \qquad (1)$$

or

$$\delta = \frac{1}{k}\ln\left[\frac{v_i}{v_{i+k}}\right] = 2\pi\xi \qquad (2)$$

- Steps
  1. Disturb the structure with an initial displacement
  2. Record motion
  3. Measure T
  4. Measure v_i and v_{i+k}
  5. Use Eq. (1) or (2) to find δ and ξ

Reference Chopra (1980)

Damped Free Vibration of a SDOF System

- *if* $c < c_{cr}$ *or*

$$\left[\frac{c}{2m}\right]^2 - \frac{k}{m} < 0$$

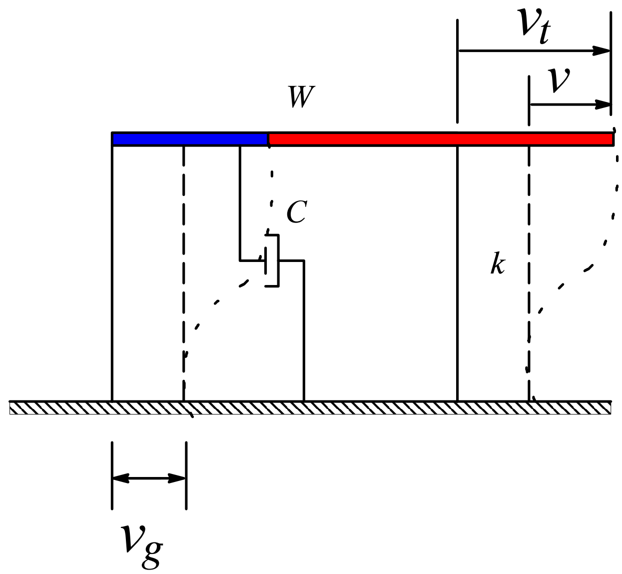
The solution to the differential equation

$$m\ddot{v} + c\dot{v} + kv = 0$$

will be

$$v(t) = e^{-\xi wt}\left[v_0 \cos\omega_D t + \frac{\dot{v}_0 + \xi\omega v_0}{\omega_D}\sin\omega_D t\right]$$

Displacement of the System Due to Ground Motion



$$v_t = v_g + v$$

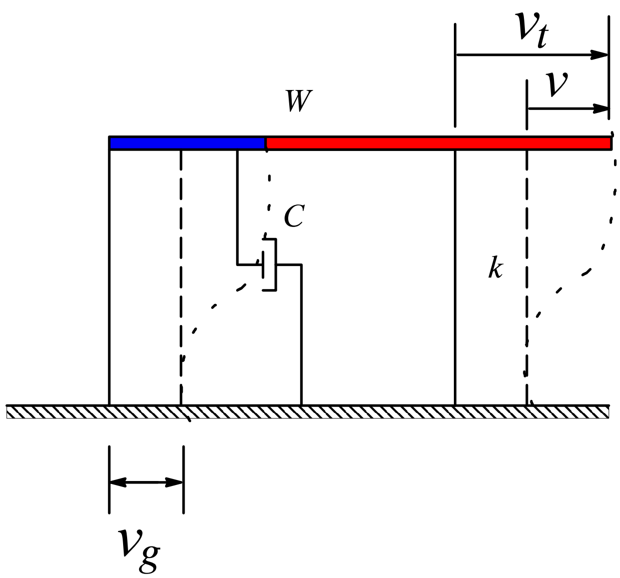
where

v_t = *Total displacement of the mass*

v_g = *Ground displacement*

v = *Relative displacement*

Equation of Motion

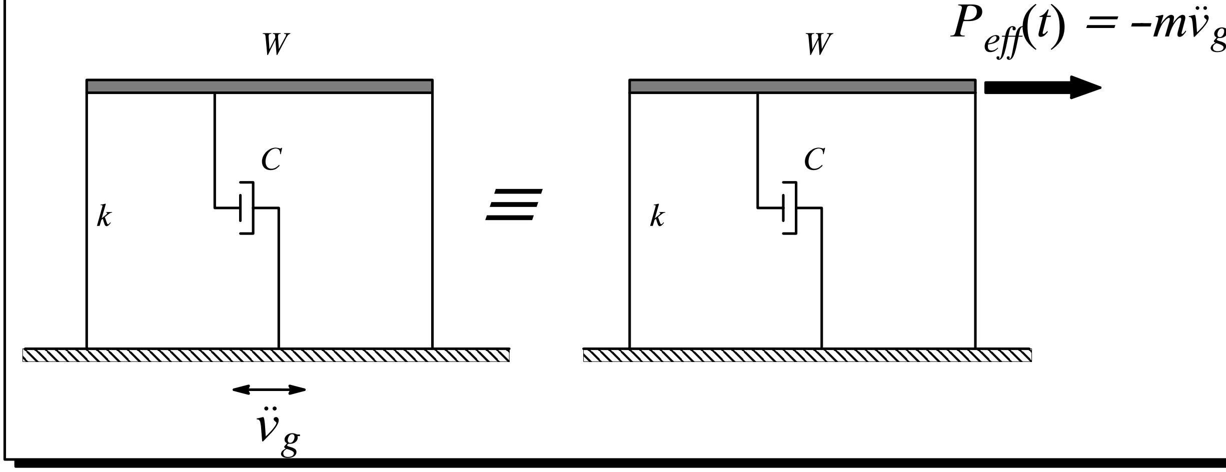


$$m(\ddot{v}_g + \ddot{v}) + c\dot{v} + kv = 0$$

where

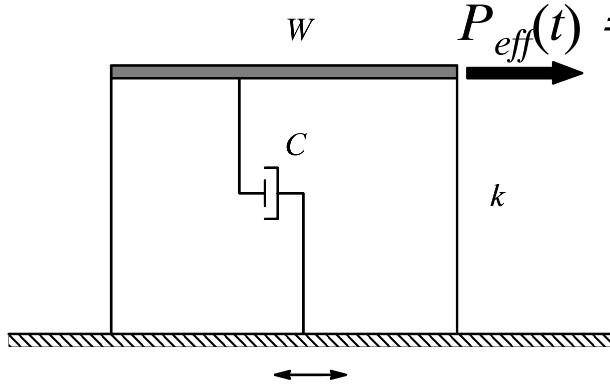
$$\ddot{v}_g = Ground\ acceleration$$

$$m\ddot{v} + c\dot{v} + kv = -m\ddot{v}_g = P_{eff}(t)$$



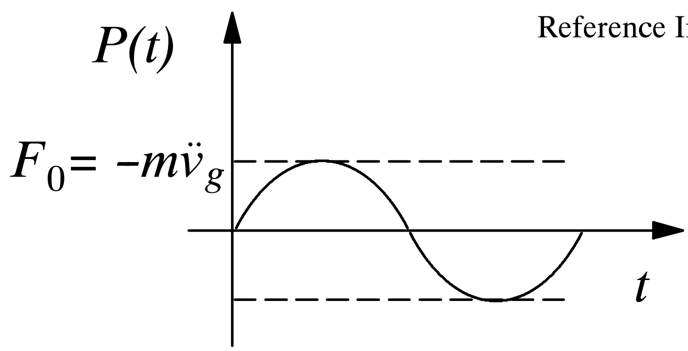
Response to Harmonic Excitation

$$P_{eff}(t) = -m\ddot{v}_g \sin\overline{\omega}t$$
$$P_{eff}(t) = F_0 \sin\overline{\omega}t$$



$$\ddot{v}_g(t) = \ddot{v}_g \sin\overline{\omega}t$$

Reference Imbsen (1995)



$$m\ddot{v} + c\dot{v} + kv = F_0 \sin\overline{\omega}t$$

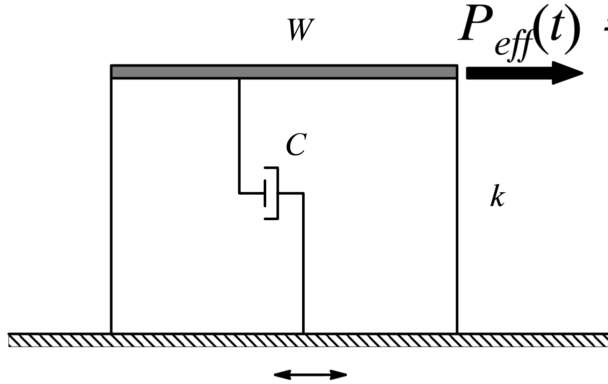
where

$$\overline{\omega} = \textit{Frequency of input motion}$$

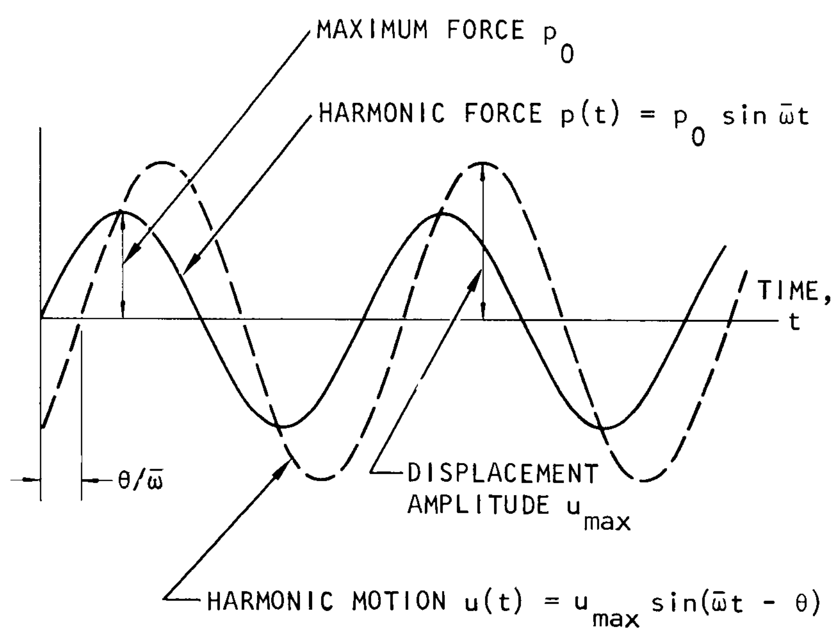
$$F_0 = \textit{Amplitude of input motion}$$

Response to Harmonic Excitation

$$P_{eff}(t) = -m\ddot{v}_g \sin\overline{\omega}t$$
$$P_{eff}(t) = F_0 \sin\overline{\omega}t$$



$$\ddot{v}_g(t) = \ddot{v}_g \sin\overline{\omega}t$$



Adopted from Chopra (1980)

Forced Vibration – Harmonic Excitation

$$m\ddot{v} + c\dot{v} + kv = -m\ddot{v}_g \sin\overline{\omega}t = F_0 \sin\overline{\omega}t$$

- Solution of Differential Equation

$$v = v_c + v_p$$

$$v_c = \textit{Complementary solution}$$

$$v_p = \textit{Particular solution}$$

- Damped system

$$v_c = e^{-\xi wt}(A\cos\omega_D t + B\sin\omega_D t)$$

Transient response – in a damped system the free vibration response of the complementary solution decays becoming insignificant

$$v_p = \frac{v_{st}\sin(\overline{\omega}t - \theta)}{\sqrt{(1 - r^2)^2 + (2\xi r)^2}}$$

Particular solution or the steady state response

Steady State Response

$$v(t) = v_{st} D \sin(\overline{\omega} t - \theta)$$

$$D = \frac{v(t)}{v_{st}} = \frac{1}{\sqrt{(1 - r^2)^2 + (2\xi r)^2}}$$

where

$$D = \textit{Dynamic magnification factor}$$

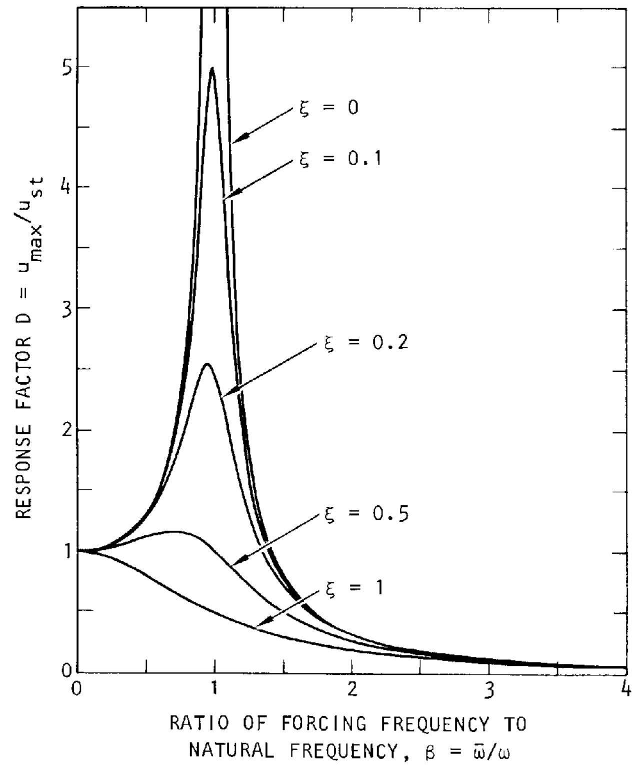
$$r = \frac{\overline{\omega}}{\omega}$$

$$v_{st} = \textit{Equivalent static displacement}$$

$$\theta = \tan^{-1}\left[\frac{2\xi r}{1 - r^2}\right]$$

Dynamic Magnification Factor

$$D = \frac{1}{\sqrt{(1 - r^2)^2 + (2\xi r)^2}}$$



Adopted Chopra (1980)

Note:

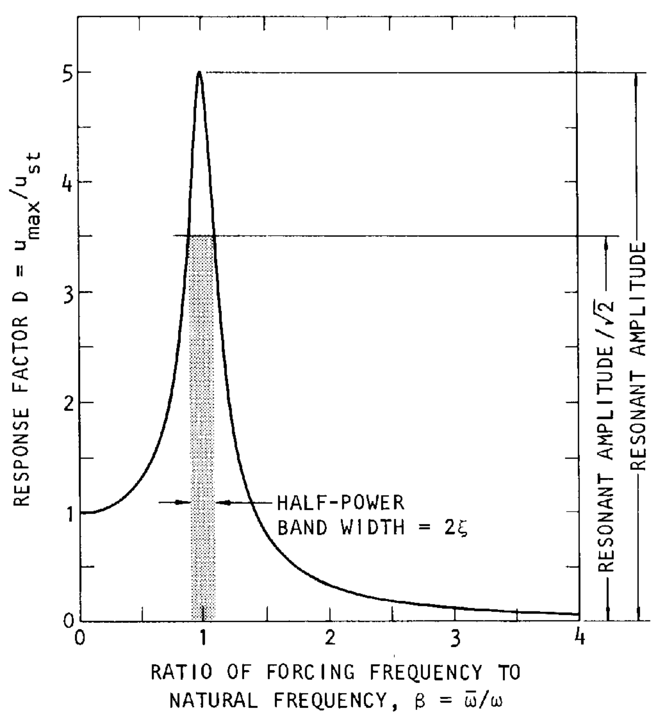
$r \rightarrow 0 \quad D \rightarrow 1 \quad \rightarrow 0 \quad v = v_{st}$

dynamic effects are negligible

$r \rightarrow 1 \quad => D = \frac{1}{2\xi}$ *Resonance*

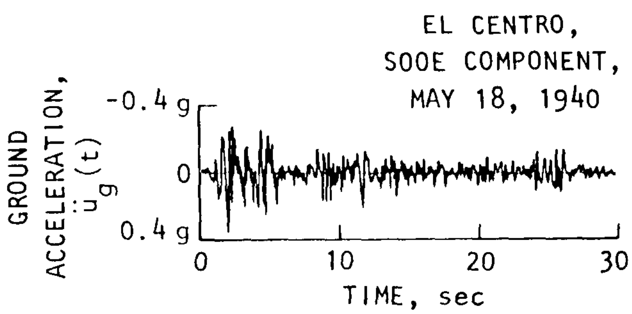
Band width (Half-Power) to Evaluate Damping

- Procedure
  1. Determine natural vibration frequency as the forcing frequency at resonance
  2. Measure half-power band width
  3. Compute ξ = (half power band width) $\times \frac{1}{2}$



Adopted Chopra (1980)

Displacement Response Spectrum



Adopted Chopra (1980)

- The response of a SDOF structure to earthquake ground motion can be written as

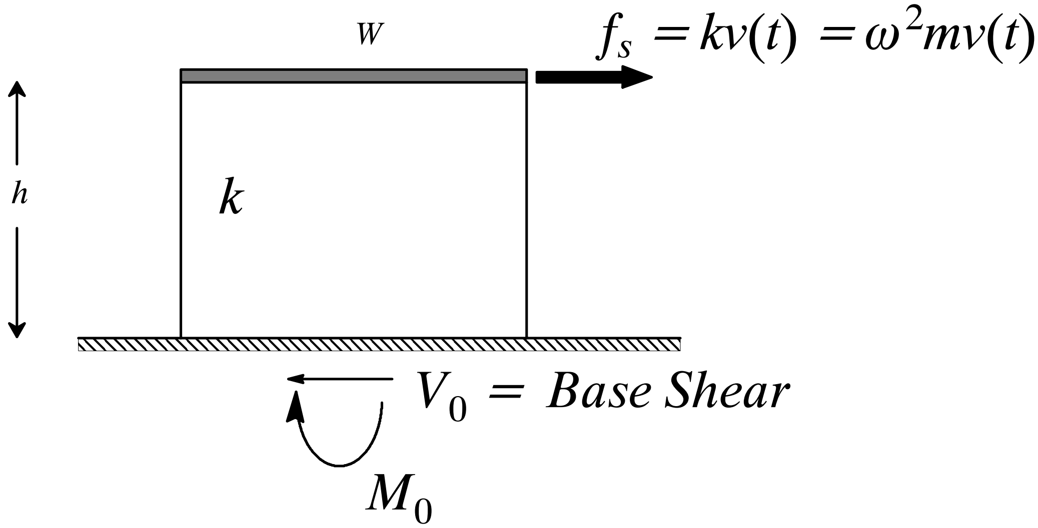
$$m\ddot{v} + c\dot{v} + kv = -m\ddot{u}_g$$

the solution

$$v(t) = -\frac{1}{\omega_D}\int_0^t \ddot{u}_g e^{-\xi\omega(t-\tau)}\sin(\omega_D(t-\tau))d\tau$$

This is called <u>Duhamel's Integral</u>

Equivalent Lateral Forces



- Static Analysis

$$V_0 = f_s$$

$$M_0 = hf_s$$

Reference Chopra (1980)

Response Spectrum

$$r_{\max} = \max_{t} |r(t)|$$

Where *r*(t) can be displacement, velocity, or acceleration.

- Response spectrum is a plot of r_{max} as a function of T or ω or f.
  - Displacement Response Spectrum

$$r = v$$
$$s_d = v_{\max}$$

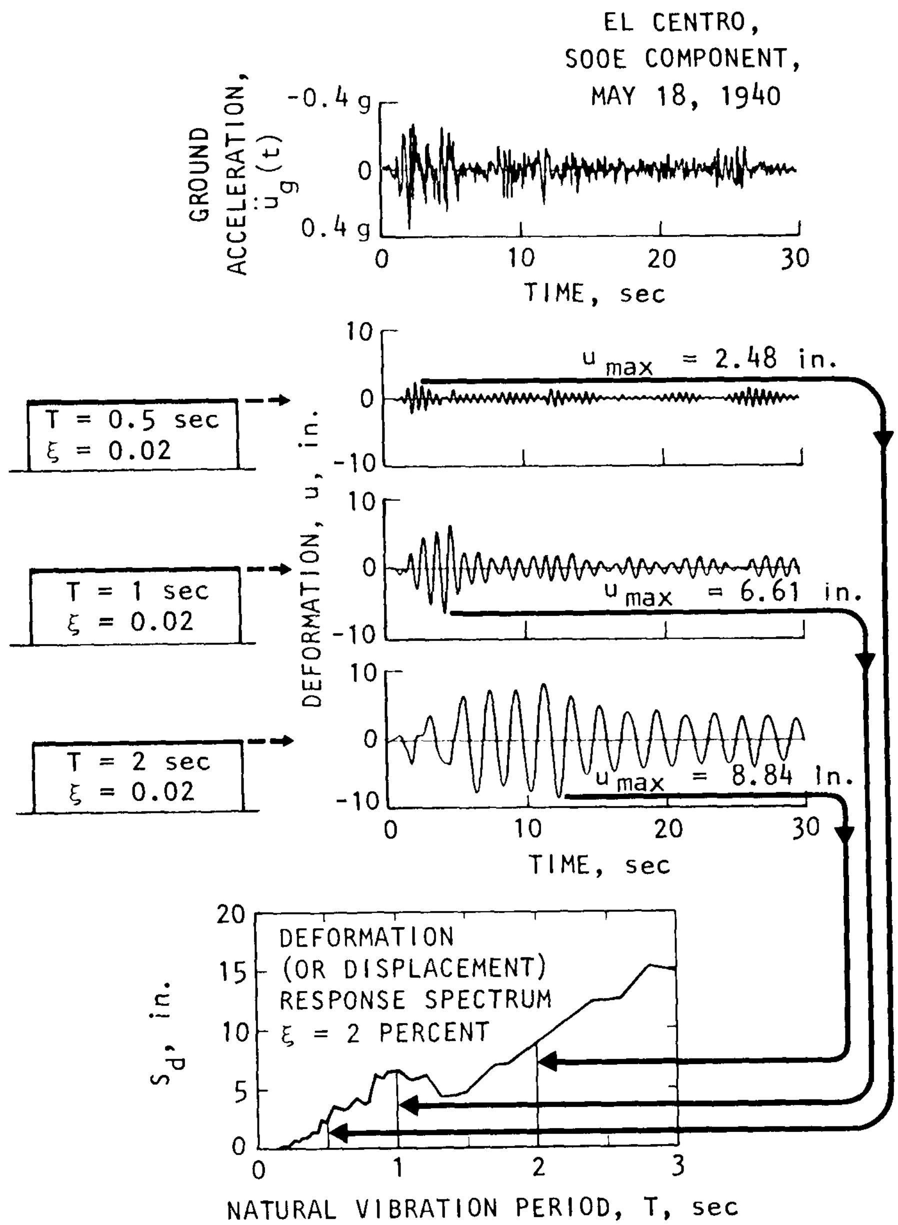
  - Psuedo-Velocity Response Spectrum

$$s_v = \omega s_d$$

  - Psuedo-Acceleration Response Spectrum

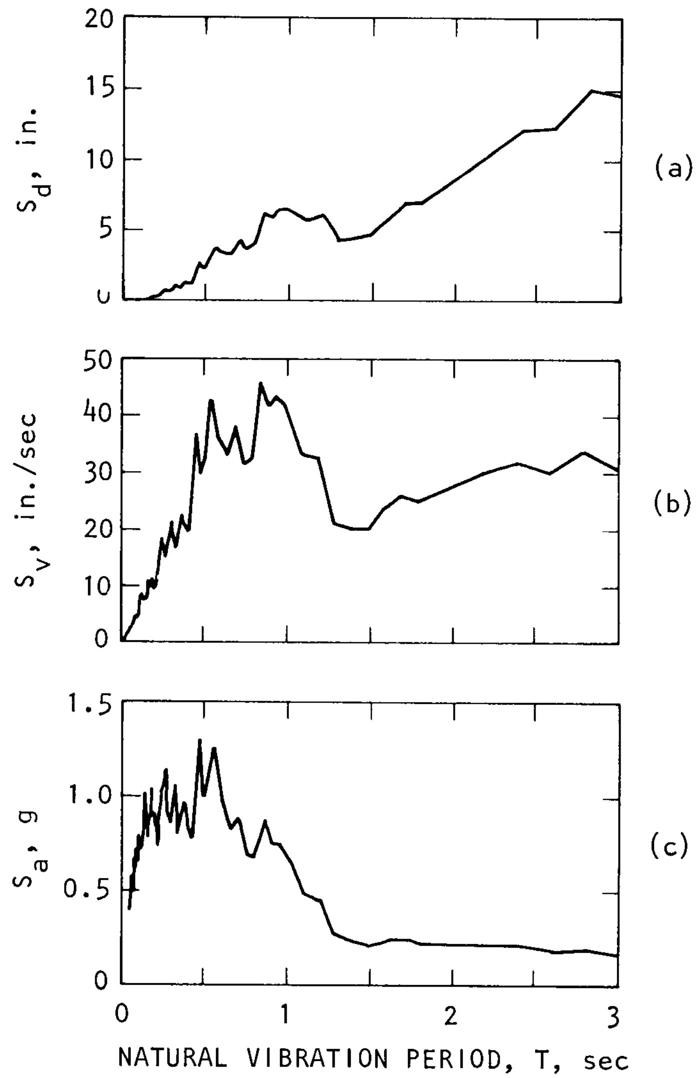
$$s_a = \omega^2 s_d = \omega s_v$$

Displacement Response Spectrum



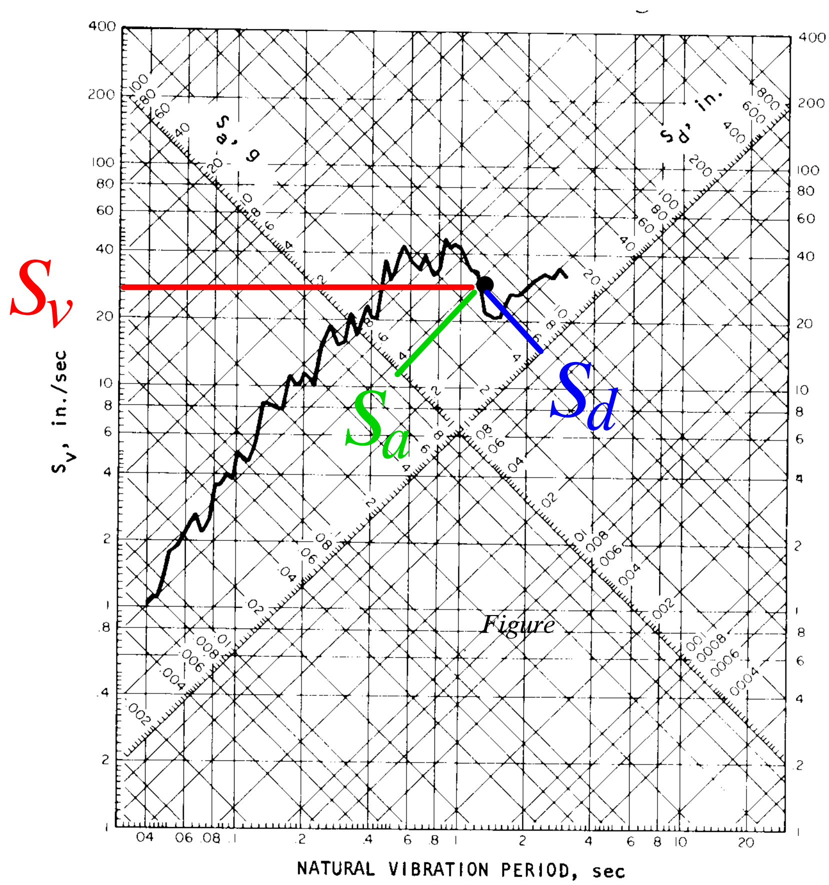
Adopted from Chopra (1980)

Displacement, Velocity, and Acceleration Response Spectrum

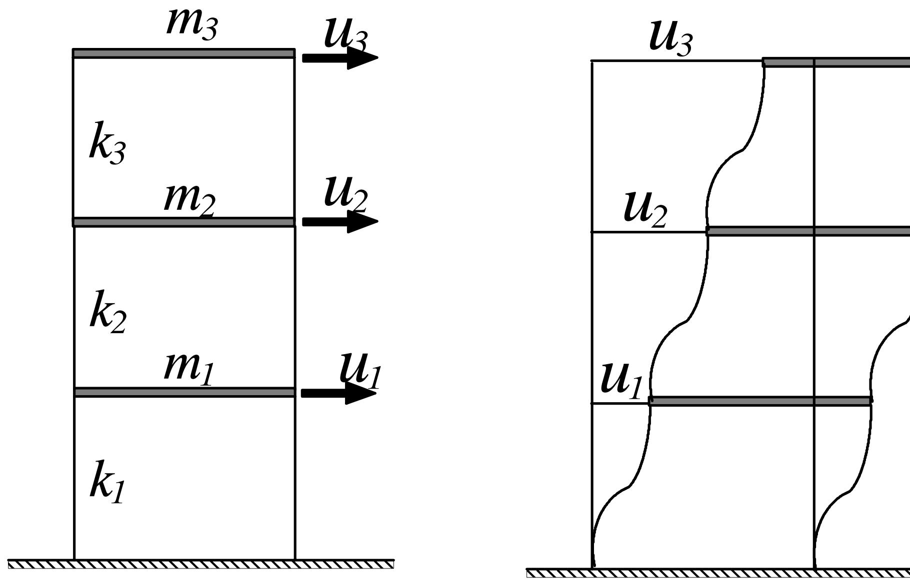


Adopted from Chopra (1980)

Response Spectrum



Multiple Degree of Freedom Structures



- Equation of motion for an undamped and lumped mass MDOF system.

$$\begin{bmatrix} k_1+k_2 & -k_2 & 0 \\ -k_2 & k_2+k_3 & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} + \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \\ \ddot{u}_3 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix}$$

$$[m]\{\ddot{u}\} + [k]\{u\} = \{F\}$$

Natural Frequency and Modes of Vibration

Assume system can vibrate in periodic manner

$$\{y(t)\}_i = \{\phi\}_i(A\sin\omega_i t + B\cos\omega_i t)$$

where

$$\{\phi\}_i = \text{ ith mode shapes}$$

$$\omega_i = \text{ ith natural frequency of vibration}$$

- Substitute in the equation of motion and simplify results in Eigen-Problem

$$[k]\{\phi\}_i = \omega_i^2[m]\{\phi\}_i$$

where

$$[k] = \text{ Stiffness Matrix}$$

$$[m] = \text{ Mass Matrix}$$

Damping
MDOF Systems

$$\omega_{iD} = \omega_i \sqrt{1 - \xi_i^2}$$

$$T_{iD} = \frac{T_i}{\sqrt{1 - \xi_i^2}}$$

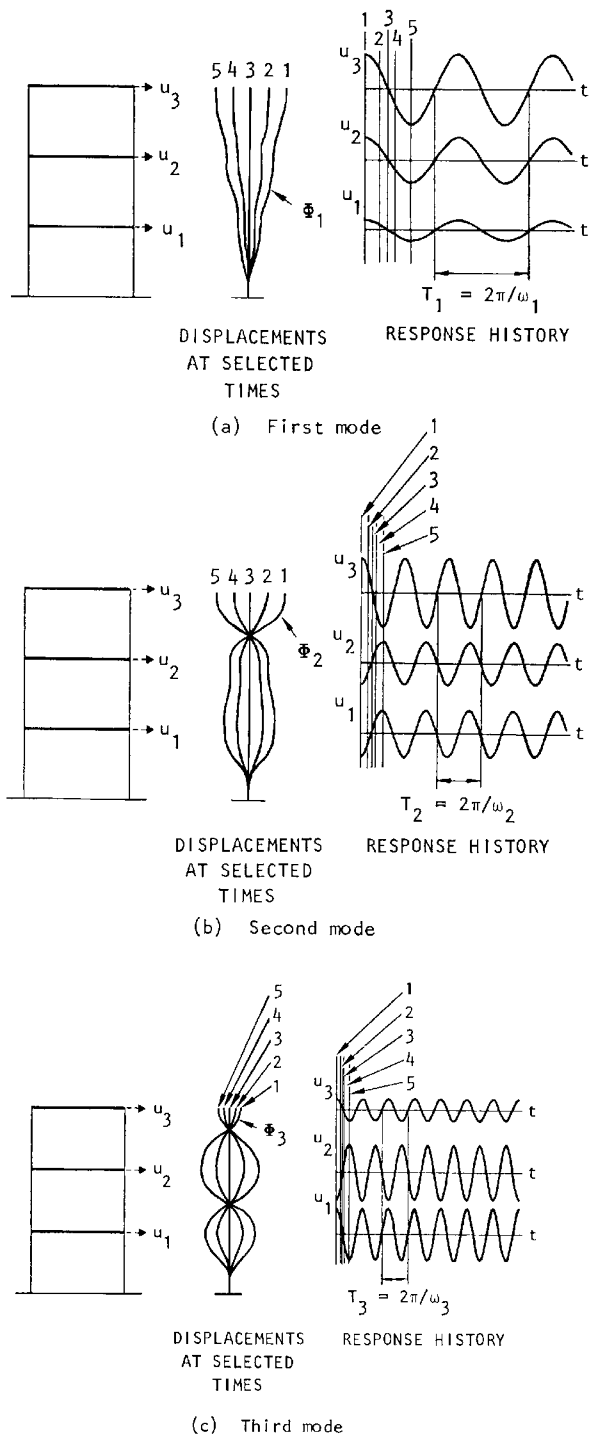
where

$\omega_{iD} =$ ith natural frequency of vibration

$T_{iD} =$ ith period of vibration of a damped system

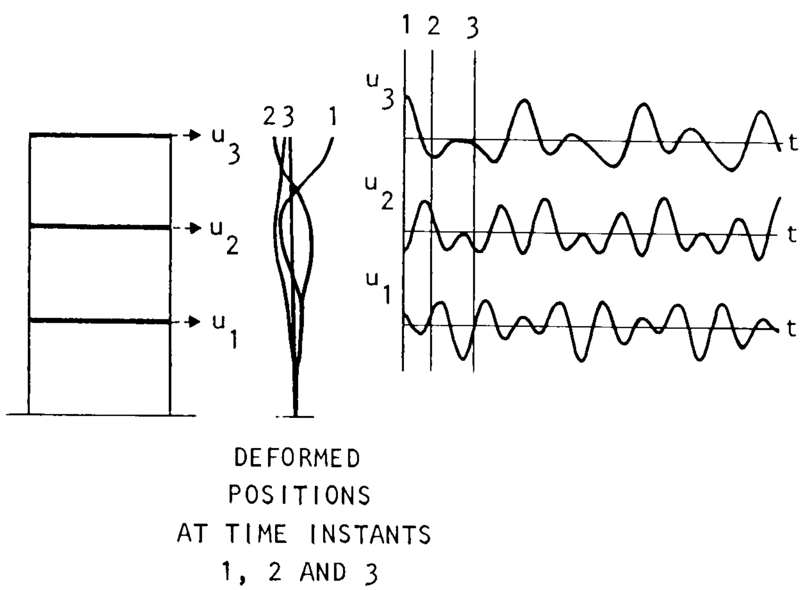
$\xi_i =$ ith mode damping ratio

Mode Shapes



Adopted from Chopra (1980)

Free Vibration Response of a MDOF System



Adopted from Chopra (1980)

Modal Superposition Method

$$[m]\{\ddot{u}\} + [c]\{\dot{u}\} + [k]\{u\} = -[m][1]\ddot{u}_g$$

where

$$[1] = \begin{Bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{Bmatrix}$$

$\ddot{u}_g =$ ground acceleration

- Coupled 2nd order ODE

Uncoupling the Equations

$$[m]\{\ddot{u}\} + [c]\{\dot{u}\} + [k]\{u\} = -[m][1]\ddot{u}_g$$

$$\{u\} = [\phi]\{Y\}$$

$$\{u\} = \sum_{i=1}^{N} Y_i\{\phi\}_i = \{\phi\}_1 Y_1 + \{\phi\}_2 Y_2 + \ \dots \ + \{\phi\}_N Y_N$$

$$\{\dot{u}\} = [\phi]\{\dot{Y}\}$$

$$\{\ddot{u}\} = [\phi]\{\ddot{Y}\}$$

where

$\{Y\} =$ Normalized coordinates

Uncoupling the Equations

$$[m]\{\ddot{u}\} + [c]\{\dot{u}\} + [k]\{u\} = -[m][1]\ddot{u}_g$$

$$\{\phi\}^T[m]\{\phi\}\{\ddot{Y}\} + \{\phi\}^T[c]\{\phi\}\{\dot{Y}\} + \{\phi\}^T[k]\{\phi\}\{Y\} = -\{\phi\}^T[m][1]\ddot{u}_g$$

- Mode Shapes are orthogonal

$$\{\phi\}_i^T[m]\{\phi\}_j = \begin{cases} M^* & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

- Triple matrix multiplications change into diagonal matrices resulting in the following uncouple differential equation

$$M_i^*\ddot{Y}_i + C_i^*\dot{Y}_i + K_i^*Y_i = -\{\phi\}_i^T[m][1]\ddot{u}_g$$

where the following are the generalized properties

$$M_i^* = \{\phi\}_i^T[m]\{\phi\}_i$$

$$C_i^* = \{\phi\}_i^T[c]\{\phi\}_i$$

$$K_i^* = \{\phi\}_i^T[k]\{\phi\}_i$$

EQUATION FOR A GIVEN MODE SHAPE

$$M_i^* \ddot{Y}_i + C_i^* \dot{Y}_i + K_i^* Y_i = -\{\phi\}_i^T [m][1] \ddot{u}_g$$

Define

$$L_i = -\{\phi\}_i^T [M][1] = \textit{Participation Factor}$$

Divide by M_i^* gives

$$\ddot{Y}_i + 2\xi\omega_i \dot{Y}_i + \omega_i^2 Y_i = -\left(\frac{L_i}{M_i^*}\right) \ddot{u}_g$$

RESPONSE SPECTRA

$$|Y|_{\max} = S_d \qquad \textit{SDOF Systems}$$

$$|Y_i|_{\max} = \left[\frac{L_i}{M_i^*}\right] S_{d_i} \qquad \textit{MDOF Systems}$$

where

$$S_{d_i} = S_{d_i}(\xi, T_i)$$

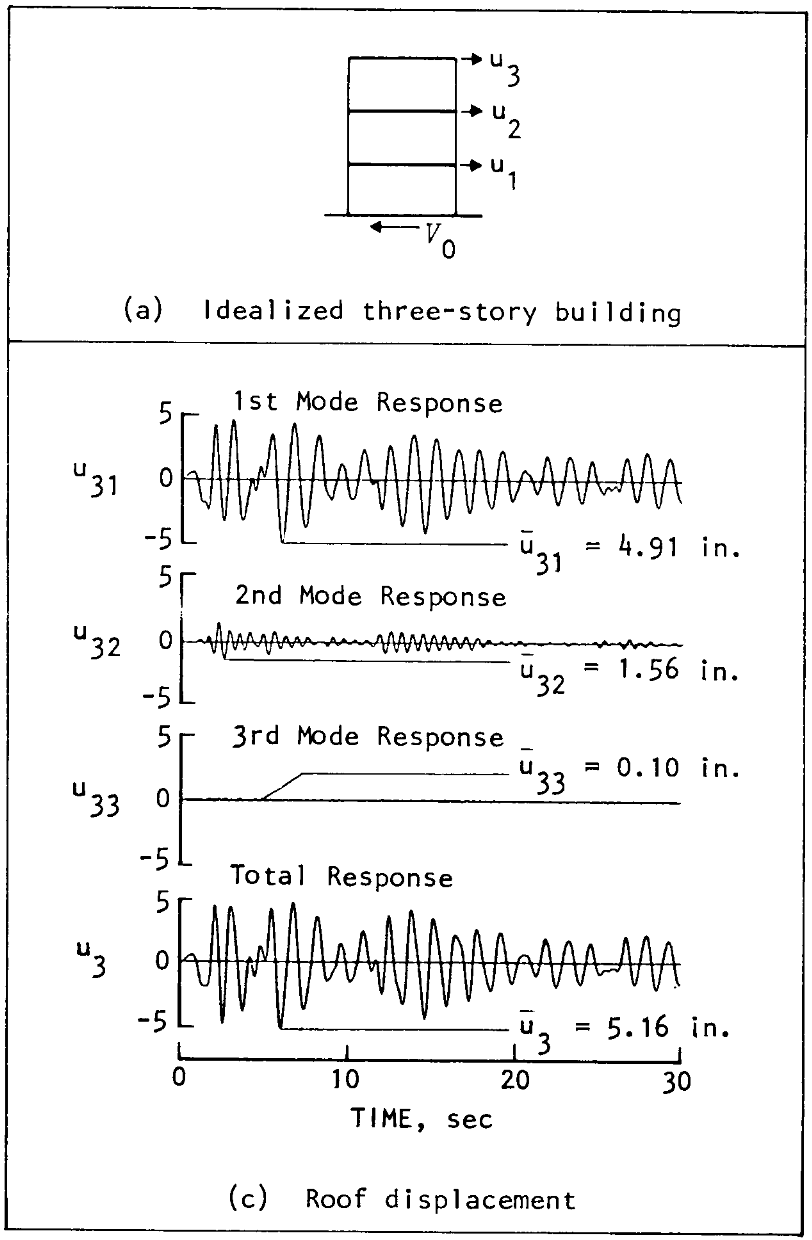
Modal contribution to the maximum response at k*th* D.O.F, (u_k) max:

$$(u_k)_{\max} = \sum_{i=1}^{n} U_{ki} = \sum_{i=1}^{n} \phi_{ki} |Y_i|_{\max} = \sum_{i=1}^{n} \phi_{ki} \left[\frac{L_i}{M_i^*}\right] S_{d_i}$$

$$f_{kn} = \frac{L_n}{M_n} \omega_n S_{vn} m_k \phi_{kn}$$

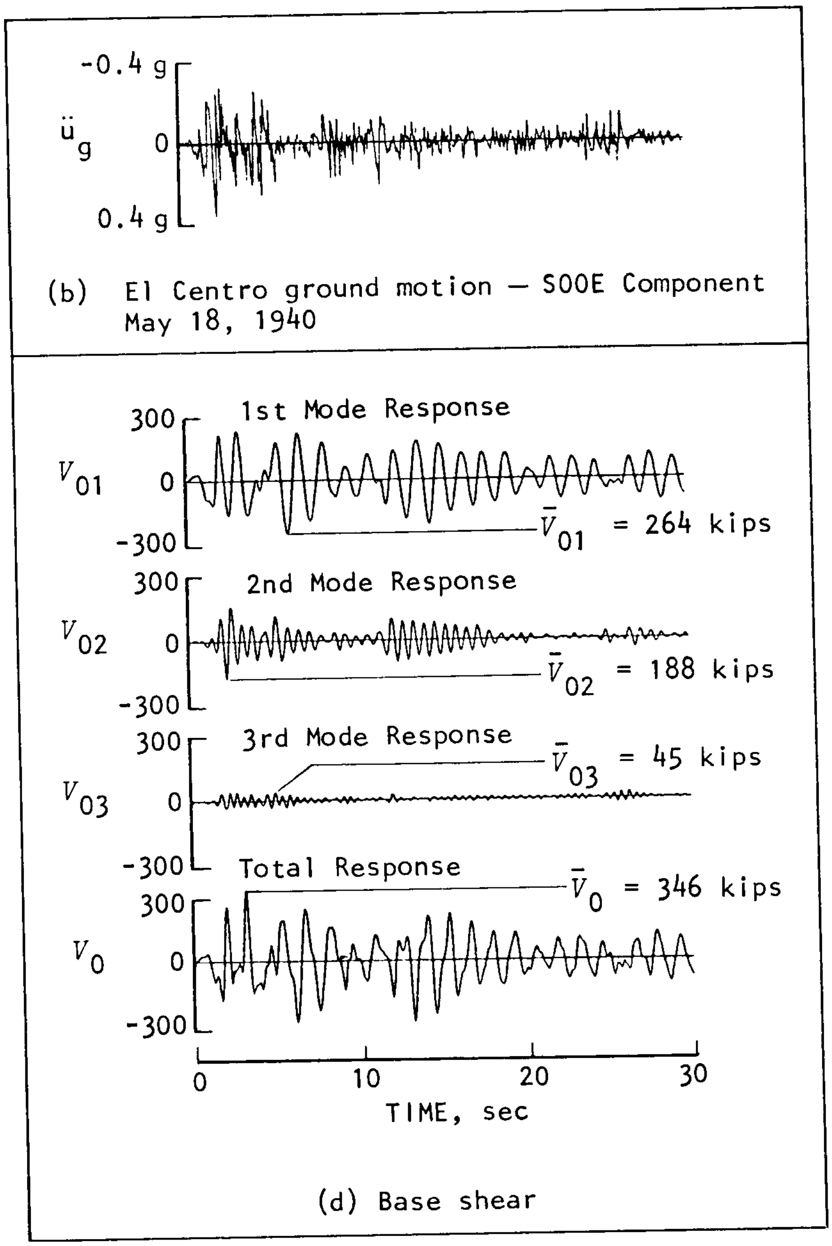
$$\{f_n\} = \frac{L_n}{M_n} \omega_n S_{vn} [M] \{\phi_n\}$$

Earthquake Response of a 3-Story Building



Adopted from Chopra (1980)

Earthquake Response of a 3-Story Building



Adopted from Chopra (1980)

Combination of Modal Response Maxima

- Combining modal responses by

  - SRSS = Square Root of the Sum of the Squares

  - CQC = Complete Quadratic Combination